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Existence of the ferromagnetic phase in competing random-bond Ising models on the square lattice for several types of probability distribution of exchange interaction

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Abstract. We generalise the arguments given in a preceding letter by the present authors to give a sufficient condition for the existence of the spontaneous magnetisation in the random-bond Ising model on the square lattice. We study in detail the system with a Gaussian distribution of exchange interaction, the system with the exchange interactions $J > 0$, 0 and $-J$, and the system with the exchange interactions $J_A > 0$ and $-J_B < 0$. It is shown that in each of these systems the ferromagnetic phase certainly exists in a region in the space of the parameters describing the probability distribution of the exchange interaction at sufficiently low temperatures. The general condition for the existence of the ferromagnetic phase is given for the system without specifying the form of the probability distribution of the exchange interaction.

1. Introduction

The quenched random-bond Ising model with competing interactions has been of great interest for years because there is a possibility that the antiferromagnetic interactions destroy the ferromagnetic long-range order and induce the spin glass state (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975, Matsubara and Sakata 1976, Domb 1976). It has been proved recently that the destruction of the ferromagnetic long-range order due to the antiferromagnetic interactions certainly happens at $p < 0.7071$ for the random-bond Ising model on the square lattice where the exchange interactions $J > 0$ and $-J$ have the respective probabilities p and $1-p$ (Horiguchi and Morita 1981). If the effects of the antiferromagnetic interactions are not so strong, the ferromagnetic order clings to the system at sufficiently low temperatures. This has been proved for the system by extending Peierls' and Griffiths' arguments (Peierls 1936, Griffiths 1964, 1972) by Avron *et al* (1981) and independently by the present authors (Horiguchi and Morita 1982a) who were stimulated by Nishimori's work (1981). The discussions have been extended to a random-site Ising model on the square lattice (Morita and Horiguchi 1982).

In the present paper, we investigate in detail the condition on the existence of the ferromagnetic phase in the random-bond Ising model on the square lattice for several types of the probability distribution of the exchange interaction. We consider the system in which the probability distribution is the Gaussian distribution. We also consider the system in which exchange interaction is assumed to take on $J > 0$, 0 and

$-J$ with respective probabilities p, r and $1-p-r$ or on $J_A>0$ and $-J_B<0$ with respective probabilities p and $1-p$. We show that in each of these systems the ferromagnetic long-range order certainly appears in a region of the space of the parameters describing the probability distribution of exchange interaction at sufficiently low temperatures. We discuss also the general condition for the existence of the ferromagnetic phase in the system without specifying any particular form for the probability distribution of the exchange interaction.

2. General formalism

In this section, we generalise the formalism given in the previous letter (Horiguchi and Morita 1982a) in which we extend Peierls' and Griffiths' arguments (Peierls 1936, Griffiths 1964, 1972) for the regular Ising model to those for the random-bond Ising model with the exchange interactions $J>0$ and $-J$. We discuss the possibility for the existence of the ferromagnetic phase in the system with the general type of the probability distribution of the exchange interaction.

We consider a random-bond Ising model on a square lattice Λ. The total number of lattice sites is denoted by N. The Hamiltonian of the system is defined by

$$H=-\sum_{\substack{(ij)\\ i,j\in\Lambda}} J_{ij}s_is_j-h\sum_{i\in\Lambda} s_i \tag{2.1}$$

where s_i is the spin variable for site i and takes on the values ±1 and h is the external field. The first summation on the right-hand side is taken over all nearest-neighbour pairs of sites, which are called bonds. J_{ij} are the exchange interactions which are quenched random variables and whose probability distribution is denoted by $P(J_{ij})$ and assumed to be independent of the J_{kl} for the other bonds (kl). We denote the configurational average of a quantity $Q\{J_{ij}\}$, which depends on the set $\{J_{ij}\}$, by angular brackets with a suffix c:

$$\langle Q\{J_{ij}\}\rangle_c=\int\ldots\int Q\{J_{ij}\}\prod_{(ij)}[P(J_{ij})\mathrm{d}J_{ij}]. \tag{2.2}$$

We assume for the exchange interaction that $\langle|J_{ij}|\rangle_c$ is finite. This condition is necessary for the existence of the thermodynamic limit of the free energy in the system (Horiguchi and Morita 1982a).

The magnetisation of the system given by (2.1) is defined by

$$\langle s_i\rangle_{h,B}=\sum_{\{s_i\}} s_i\,\mathrm{e}^{-\beta H}\Big/\sum_{\{s_i\}}\mathrm{e}^{-\beta H} \tag{2.3}$$

where $\beta=1/kT$ as usual and k is Boltzmann's constant. Here the suffix B means a boundary condition imposed on the system. We consider the boundary condition B_0 that the boundary spins are not coupled with the outer system and the one B_1 that the boundary spins are forced to align upwards. We then have the following inequality (Horiguchi and Morita 1982a)

$$m_s\geqslant m_{B_1}(h=0) \tag{2.4}$$

where

$$m_s = \lim_{h\to+0} \lim_{N\to\infty} \frac{1}{N} \sum_{i\in\Lambda} \langle\langle s_i \rangle_{h,B_0}\rangle_c \tag{2.5}$$

and

$$m_{B_1}(h=0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i\in\Lambda} \langle\langle s_i \rangle_{0,B_1}\rangle_c. \tag{2.6}$$

Equation (2.6) is expressed as follows

$$m_{B_1}(h=0) = 1 - 2 \lim_{N\to\infty} \frac{1}{N} \langle\langle N_- \rangle_{0,B_1}\rangle_c \tag{2.7}$$

where N_- is the total number of sites with down spins in the system. When we draw lines bisecting nearest-neighbour pairs of sites i and j when s_i is $+1$ on the site i and s_j is -1 on the other site j in a spin configuration $\{s_i\}$ of the system, these lines form closed polygons under the boundary condition B_1 (Peierls 1936, Griffiths 1964, 1972). We introduce a function $X_{2n}^{(l)}$ which takes the value 1 if the lth polygon of perimeter $2n$ occurs in a spin configuration and 0 otherwise. Then we have

$$N_- < \sum_{n=2,3,\ldots} \frac{n^2}{4} \sum_{l=1}^{\nu(2n)} X_{2n}^{(l)}. \tag{2.8}$$

$\nu(2n)$ is the number of different polygons of perimeter $2n$ in the system and is estimated as follows (Griffiths and Lebowitz 1968, Horiguchi and Morita 1982a):

$$\nu(2n) \leq (9^{n-1}/n)N. \tag{2.9}$$

The thermal average of $X_{2n}^{(l)}$ is expressed as

$$\langle X_{2n}^{(l)} \rangle_{0,B_1} = \sum_{\{s_i\}}{}' e^{-\beta H_1} \Big/ \sum_{\{s_i\}} e^{-\beta H_1} \tag{2.10}$$

where

$$H_1 = -\sum_{\substack{(ij)\\ i,j\in\Lambda\backslash\bar{\Lambda}}} J_{ij} s_i s_j - \sum_{\substack{(ij)\\ i\in\Lambda\backslash\bar{\Lambda}\\ j\in\bar{\Lambda}}} J_{ij} s_i. \tag{2.11}$$

Here $\bar{\Lambda}$ is the set of sites j which are on the boundary of the lattice Λ. The sum in the denominator is taken over all the spin configurations satisfying the boundary condition, but the sum in the numerator only over all the configurations in which the lth polygon of perimeter $2n$ appears. We restrict the sum in the denominator to the configurations that appear in the numerator and to those that are generated from these spin configurations by reversing all the spins inside the polygon. Then we have an upper bound to $\langle X_{2n}^{(l)} \rangle_{0,B_1}$. After taking the configurational average, it reads

$$\langle\langle X_{2n}^{(l)} \rangle_{0,B_1}\rangle_c \leq \left\langle \left[1 + \exp\left(2\beta \sum_{(ij)\in\mathscr{L}(2n)} J_{ij} \right) \right]^{-1} \right\rangle_c \tag{2.12}$$

where $\mathscr{L}(2n)$ is the set of nearest-neighbour pairs of sites (ij) which are situated across the periphery of the polygon of perimeter $2n$. The right-hand side of (2.12) does not depend on the suffix l any more. We have from equation (2.8)

$$\lim_{N\to\infty} \frac{1}{N} \langle\langle N_- \rangle_{0,B_1}\rangle_c \leq \frac{1}{36} \sum_{n=2}^{\infty} n 9^n \left\langle \left[1 + \exp\left(2\beta \sum_{(ij)\in\mathscr{L}(2n)} J_{ij} \right) \right]^{-1} \right\rangle_c. \tag{2.13}$$

As a sufficient condition for the existence of the ferromagnetic phase we have that the right-hand side of equation (2.13) is less than $\frac{1}{2}$.

We overestimate $\langle X_{2n}^{(l)}\rangle_{0,B_1}$ by 1 when the sum of the exchange interactions along the periphery of the polygon is negative and by $\exp(-2\beta\sum_{(ij)\in\mathscr{L}(2n)}J_{ij})$ otherwise. Then we have

$$\lim_{N\to\infty}\frac{1}{N}\langle\langle N_-\rangle_{0,B_1}\rangle_c \leqslant \frac{1}{36}\sum_{n=2}^{\infty} n9^n \underset{\sum_{(ij)\in\mathscr{L}(2n)} J_{ij}<0}{\int\cdots\int} \prod_{(ij)}\{P(J_{ij})\mathrm{d}J_{ij}\} + \frac{1}{36}\sum_{n=2}^{\infty} n9^n \underset{\sum_{(ij)\in\mathscr{L}(2n)} J_{ij}\geqslant 0}{\int\cdots\int} \exp\left(-2\beta\sum_{(ij)\in\mathscr{L}(2n)}J_{ij}\right)\prod_{(ij)}\{P(J_{ij})\mathrm{d}J_{ij}\}. \tag{2.14}$$

The first series on the right-hand side of equation (2.14) converges when the following inequality is satisfied

$$f(t_0)\equiv\langle\exp(-t_0J_{ij})\rangle_c<\tfrac{1}{3} \tag{2.15}$$

when t_0 is a positive solution of the equation

$$\langle J_{ij}\exp(-t_0J_{ij})\rangle_c=0. \tag{2.16}$$

When equation (2.16) does not have a positive solution, the first term on the right-hand side of equation (2.14) never converges. We notice here that the function $f(t)$ is a convex function and there is only one real solution of equation (2.16) if it exists. When the probability distribution $P(J_{ij})$ takes the form $\delta(J_{ij}-J_0)$ with $J_0>0$ in some limiting values of the parameters describing the probability distribution, there is a range of parameters in which the ferromagnetic state certainly occurs in the ground state.

The second series on the right-hand side of equation (2.14) converges under the condition

$$\langle\exp[(t_1-2\beta)J_{ij}]\rangle_c<\tfrac{1}{3} \tag{2.17}$$

when t_1 is a positive solution of the equation

$$\langle J_{ij}\exp[(t_1-2\beta)J_{ij}]\rangle_c=0 \tag{2.18}$$

or under the condition

$$\langle\exp(-2\beta J_{ij})\rangle_c<\tfrac{1}{3} \tag{2.19}$$

when equation (2.18) has no positive solution. The condition (2.17) is equivalent to the one (2.15) for $t_0\leqslant 2\beta$ and the condition (2.19) includes the one (2.15) for $t_0\geqslant 2\beta$. Thus at finite temperatures, the right-hand side of equation (2.14) converges under the conditions that either equation (2.15) for $t_0\leqslant 2\beta$ or equation (2.19) for $t_0\geqslant 2\beta$ is satisfied. When the probability distribution is given by $\delta(J_{ij}-J_0)$ with $J_0>0$, equation (2.16) has a solution $t_0=+\infty$ which is greater than 2β. Then (2.19) applies, that is equation (2.14) converges when $kT/J_0<2/\log 3$ (Griffiths 1964).

We investigate in detail the condition that equation (2.13) or (2.14) is less than $\frac{1}{2}$ for several types of the probability distribution $P(J_{ij})$ in the following sections.

3. Gaussian distribution

We consider the system (2.1) in which the probability distribution of the exchange interaction is the Gaussian distribution

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(J_{ij}-\bar{J})^2\right) \tag{3.1}$$

where $\bar{J}$ is the mean and σ is the standard deviation. We assume $\bar{J}>0$ and set $x=\bar{J}/\sigma$. This system was studied by Klein *et al* (1979) in terms of the self-consistent mean-random-field approximation. An exact lower bound to the critical ratio x_c below which there is no spontaneous magnetisation was given by the present authors (Horiguchi and Morita 1982b, c). Its value is 0.494 for the square lattice.

From equation (2.12), we have

$$\langle\langle X_{2n}^{(l)}\rangle_{0,B_1}\rangle_c \leqslant \int\ldots\int\left[1+\exp\left(2\sum_{(ij)\in\mathscr{L}(2n)} J_{ij}\right)\right]^{-1}\prod_{(ij)}\{P(J_{ij})\mathrm{d}J_{ij}\} \tag{3.2}$$

$$= \left(\frac{1}{\sqrt{\pi}}\right)^{2n}\int\ldots\int\exp\left(-\sum_{(ij)\in\mathscr{L}(2n)} t_{ij}^2\right)$$
$$\times\left\{1+\exp\left[\alpha\left(\sum_{(ij)\in\mathscr{L}(2n)} t_{ij}+2n\gamma\right)\right]\right\}^{-1}\prod_{(ij)}\mathrm{d}t_{ij} \tag{3.3}$$

where we put $\alpha=2\sqrt{2}\sigma\beta$ and $\gamma=\bar{J}/\sqrt{2}\sigma$. We make in equation (3.3) a transformation from the set of variables $\{t_{ij}|(ij)\in\mathscr{L}(2n)\}$ to the set of variables $\{u_j|j=1,2,\ldots,2n\}$ under the conditions that $\Sigma\, t_{ij}=\sqrt{2n}\,u_1$, $\Sigma\, t_{ij}^2=\Sigma\, u_j^2$ and the Jacobian of the transformation is equal to 1. Then we carry out the integration in (3.3) except the one for u_1 and we have

$$\langle\langle X_{2n}^{(l)}\rangle_{0,B_1}\rangle_c \leqslant \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-u_1^2}\{1+\exp[\alpha(\sqrt{2n}u_1+2n\gamma)]\}^{-1}\,\mathrm{d}u_1. \tag{3.4}$$

We overestimate the second factor of the integrand by 1 when $u_1<-\sqrt{2n}\gamma$ and by $\exp[-\alpha(\sqrt{2n}u_1+2n\gamma)]$ when $u_1\geqslant-\sqrt{2n}\gamma$ and then we have

$$\langle\langle X_{2n}^{(l)}\rangle_{0,B_1}\rangle_c \leqslant \tfrac{1}{2}\operatorname{erfc}(\sqrt{n}x)+\tfrac{1}{2}\exp[4n(1/y^2-x/y)]\operatorname{erfc}[\sqrt{n}(2/y-x)] \tag{3.5}$$

where $y=1/\sigma\beta$ and $\operatorname{erfc}(z)$ is the error function (Magnus *et al* 1966). Then we have

$$\lim_{N\to\infty}\frac{1}{N}\langle\langle N_-\rangle_{0,B_1}\rangle_c \leqslant \tfrac{1}{72}\sum_{n=2}^{\infty} n9^n\operatorname{erfc}(\sqrt{n}x)$$
$$+\tfrac{1}{72}\sum_{n=2}^{\infty} n9^n\exp[4n(1/y^2-x/y)]\operatorname{erfc}[\sqrt{n}(2/y-x)]. \tag{3.6}$$

The above equation is also obtained from equation (2.14) by using successively the identity

$$\int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{-z^2}\operatorname{erfc}(az+b)=\sqrt{\pi}\operatorname{erfc}[b/(a^2+1)^{1/2}] \tag{3.7}$$

or by using the same transformation of variables leading to equation (3.4) from equation (3.3).

We have the following inequality for the error function

$$\frac{2z^2-1}{2\sqrt{\pi}z^3}\,e^{-z^2}<\operatorname{erfc}(z)<\frac{1}{\sqrt{\pi}z}\,e^{-z^2}. \tag{3.8}$$

By using this inequality, we have the condition that each series in equation (3.6) converges. At $T=0$, we find the condition $x>\sqrt{2\log 3}$. At finite temperatures, the right-hand side of equation (3.6) converges for the union of the sets of points $\{x, y|xy\leqslant 2, x>\sqrt{2\log 3}\}$ and $\{x, y|xy\geqslant 2, y^2\log 3-2xy+2<0\}$. These conditions are also obtained from equation (2.15) for $t_0\leqslant 2\beta$ and equation (2.19) for $t_0\geqslant 2\beta$ where $t_0=x/\sigma$.

We perform numerical calculations to find the region in the $\bar{J}/\sigma-kT/\sigma$ plane where the ferromagnetic phase certainly occurs. We investigate the condition that the right-hand side of (3.6) is less than $\frac{1}{2}$ and show the boundary by the bold full curve in figure 1. There certainly exists the ferromagnetic phase to the right of the line. The ferromagnetic ground state occurs for $x>1.49$. The light full curve was obtained in the previous papers (Horiguchi and Morita 1982b, c) and shows that there is no spontaneous magnetisation to the left of the line. The broken line shows the Curie temperature in the molecular field approximation for the regular Ising model with the exchange interaction $\bar{J}>0$.

4. Discrete distribution

In this section, we consider the system in which the exchange interaction takes on the values $J_A>0$, 0 and $-J_B<0$ with respective probabilities p, r, and $1-p-r$:

$$P(J_{ij})=p\delta(J_{ij}-J_A)+r\delta(J_{ij})+(1-p-r)\delta(J_{ij}+J_B). \tag{4.1}$$

In the case of $J_A=J_B=J$, the system was studied by several authors in order to investigate the effects of percolation on spin glass phase (e.g. Giri and Stephen 1978, Southern *et al* 1979 etc). The present authors showed that there is no spontaneous magnetisation at $T=0$ when the following inequality is satisfied (Horiguchi and Morita 1981, 1982b)

$$p<(1-r)/[1+(\sqrt{2}-1)^{1/(1-r)}]. \tag{4.2}$$

In the case of $r=0$ in equation (4.1), the system was studied in the Bethe approximation (e.g. Matsubara and Sakata 1976, Katsura *et al* 1979 etc).

We assume that a polygon of perimeter $2n$ is made of lines bisecting l bonds with the exchange interaction J_A, m bonds with the zero exchange interaction and $2n-l-m$ bonds with $-J_B$. We then have from equation (2.13)

$$\lim_{N\to\infty}\frac{1}{N}\langle\langle N_-\rangle_{0,B_1}\rangle_c\leqslant\frac{1}{36}\sum_{n=2}^{\infty}n9^n\sum_{m=0}^{2n}\sum_{l=0}^{2n-m} \times\frac{(2n)!p^lr^m(1-p-r)^{2n-l-m}}{l!m!(2n-l-m)!\{1+\exp[2\beta(lJ_A-(2n-l-m)J_B)]\}}. \tag{4.3}$$

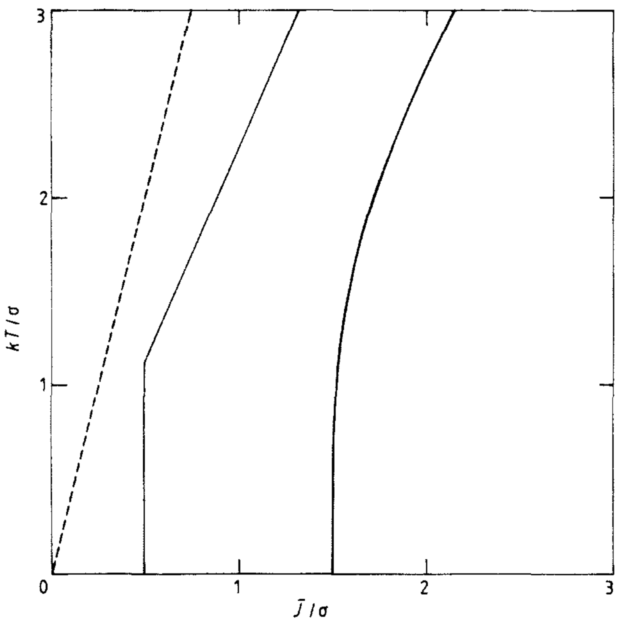

Figure 1. The random-bond Ising model on the square lattice with Gaussian distribution. The right of the bold full curve is the region where spontaneous magnetisation exists. The left of the light full curve is the region where there is no spontaneous magnetisation. The broken line indicates the Curie temperature in the molecular field approximation for the ferromagnetic Ising model with the exchange interaction $\bar{J}$.

In the case of $J_A = J_B = J$, we have from (2.14)

$$\lim_{N\to\infty}\frac{1}{N}\langle\langle N_-\rangle_{0,B_1}\rangle_c$$
$$\leqslant\frac{1}{36}\sum_{n=2}^{\infty} n9^n(1-p-r)^{2n}\sum_{m=0}^{2n}\sum_{l=0}^{l_0}\frac{(2n)!}{l!\,m!\,(2n-l-m)!}\left(\frac{p}{1-p-r}\right)^l\left(\frac{r}{1-p-r}\right)^m$$
$$+\frac{1}{36}\sum_{n=2}^{\infty} n9^n\left(\frac{1-p-r}{e^{-2\beta J}}\right)^{2n}\sum_{m=0}^{2n}\sum_{l=l_0+1}^{2n-m}\frac{(2n)!}{l!\,m!\,(2n-l-m)!}$$
$$\times\left(\frac{p\,e^{-4\beta J}}{1-p-r}\right)^l\left(\frac{r\,e^{-4\beta J}}{1-p-r}\right)^m \tag{4.4}$$

where $l_0=[n-m/2]$; $[a]$ denotes the greatest integer not greater than a. By using Cauchy's test for convergence, we have that the right-hand side of equation (4.4) converges under the conditions

$$p\leqslant(1-r)/(1+e^{-4\beta J}) \qquad \text{and} \qquad p>\tfrac{1}{2}(1-r)+\tfrac{1}{3}\sqrt{2-3r} \tag{4.5}$$

or

$$1-r\geqslant p\geqslant(1-r)/(1+e^{-4\beta J}) \qquad \text{and} \qquad p>[1-r+e^{-2\beta J}(r-\tfrac{1}{3})]/(1-e^{-4\beta J}) \tag{4.6}$$

where $0 \leqslant r < \frac{1}{3}$. In the case of $r = 0$, we have from (2.14)

$$\lim_{N\to\infty} \frac{1}{N} \langle\langle N_-\rangle_{0,B_1}\rangle_c \leqslant \frac{1}{36} \sum_{n=2}^{\infty} n 9^n (1-p)^{2n} \sum_{l=0}^{l_0} \frac{(2n)!}{l!(2n-l)!} \left(\frac{p}{1-p}\right)^l + \frac{1}{36} \sum_{n=2}^{\infty} n 9^n \left(\frac{1-p}{e^{-2\beta J_B}}\right)^{2n} \sum_{l=l_0+1}^{2n} \frac{(2n)!}{l!(2n-l)!} \left(\frac{p \exp[-2\beta(J_A+J_B)]}{1-p}\right)^l \tag{4.7}$$

where $l_0 = [2nJ_B/(J_A+J_B)]$. The right-hand side of equation (4.7) converges by Cauchy's test under the conditions

$$p \leqslant J_B/\{J_B + J_A \exp[-2\beta(J_A+J_B)]\} \qquad \text{and} \qquad [p(1+J_A/J_B)]^{J_B/(J_A+J_B)}[(1-p)(1+J_B/J_A)]^{J_A/(J_A+J_B)} < \tfrac{1}{3} \tag{4.8}$$

or

$$p \geqslant J_B/\{J_B + J_A \exp[-2\beta(J_A+J_B)]\} \qquad \text{and} \qquad p > (1 - \tfrac{1}{3} e^{-2\beta J_B})/\{1 - \exp[-2\beta(J_A+J_B)]\}. \tag{4.9}$$

These conditions (4.5) and (4.6), and (4.8) and (4.9) are also obtained from the general conditions given in § 2. The general convergence conditions for the probability distribution (4.1) are obtained from those inequalities (2.15) for $t_0 \leqslant 2\beta$, and (2.19) for $t_0 \geqslant 2\beta$ where $t_0 = \log\{pJ_A/(1-p-r)J_B\}/(J_A+J_B)$.

The numerical calculations of finding the region that the right-hand side of (4.3) is less than $\frac{1}{2}$ are performed for the case of $J_A = J_B = J$ and for the case of $r = 0$, respectively. The results obtained are given by bold full curves in figure 2 for the case of $J_A = J_B = J$ and in figure 3 for the case of $r = 0$. On the lower-temperature side of the full curves in these figures, the spontaneous magnetisation certainly exists in each case. In figure 2, we also give the results obtained in the previous papers (Horiguchi and Morita 1981, 1982b) by the light full curve for each value of r which consists of two straight lines $kT/J = \frac{1}{2}\log(\sqrt{2}-1)$ and $p = (1-r)/[1+(\sqrt{2}-1)^{1/(1-r)}]$. There is no spontaneous magnetisation on the higher-temperature side and lower-p side of these lines.

5. Concluding remarks

We proved that the ferromagnetic state certainly appears in a region in the space of parameters describing the probability distribution of exchange interaction at sufficiently low temperatures for the random-bond Ising model on the square lattice when the probability distribution is given by a Gaussian distribution, the discrete distribution with $J > 0$, 0 and $-J$ or the one with $J_A > 0$ and $-J_B < 0$. The results can be extended to the case of the antiferromagnetic state. We divide the lattice sites into two sublattices, replace the spin variables s_i on the sites belonging to one of the sublattices by $-s_i$ and invert the direction of the external field on these sites. Then we can show that the system is the antiferromagnetic state if the system is ferromagnetic when $\bar{J}/\sigma < 0$ is replaced by $|\bar{J}|/\sigma$ for the Gaussian distribution and when the concentration of the ferromagnetic interaction is equal to $1-p-r$ for the discrete distribution.

We are able to extend the present results to the systems on other two-dimensional lattices. The number $\nu(b)$ of polygons of perimeter b is overestimated by

$$\nu(b) \leqslant [z(z-1)^{b-2}/2b]N_d \tag{5.1}$$

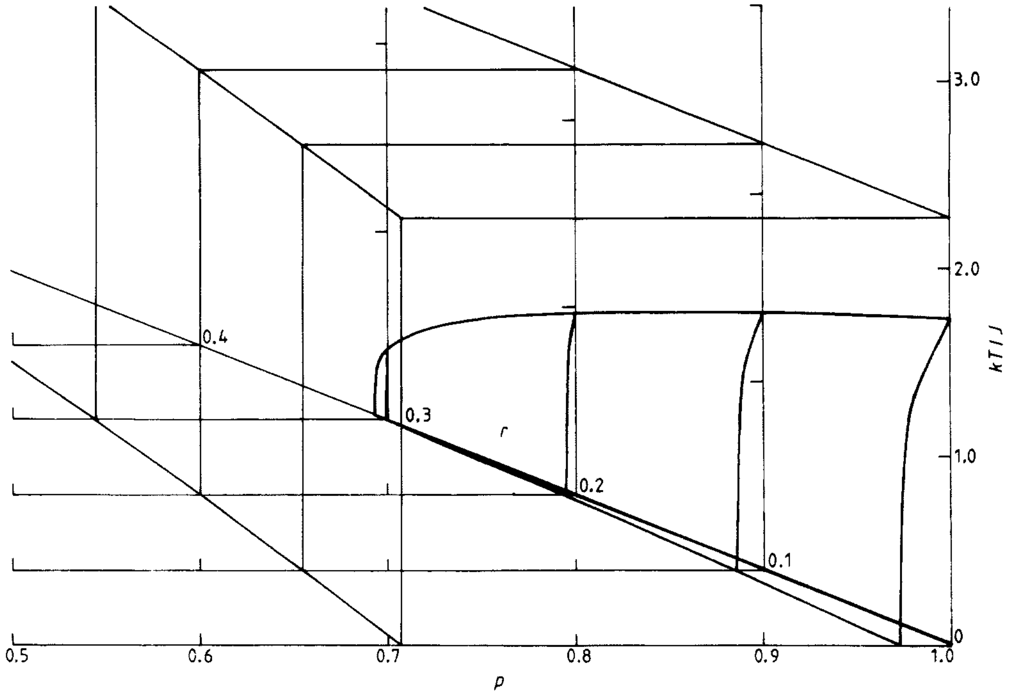

Figure 2. The random-bond Ising model on the square lattice with exchange interaction $J>0$, 0 and $-J$ with respective probabilities p, r and $1-p-r$. The lower temperature side of the bold full curves shows that spontaneous magnetisation exists. On the outside of the light full lines, there is no spontaneous magnetisation.

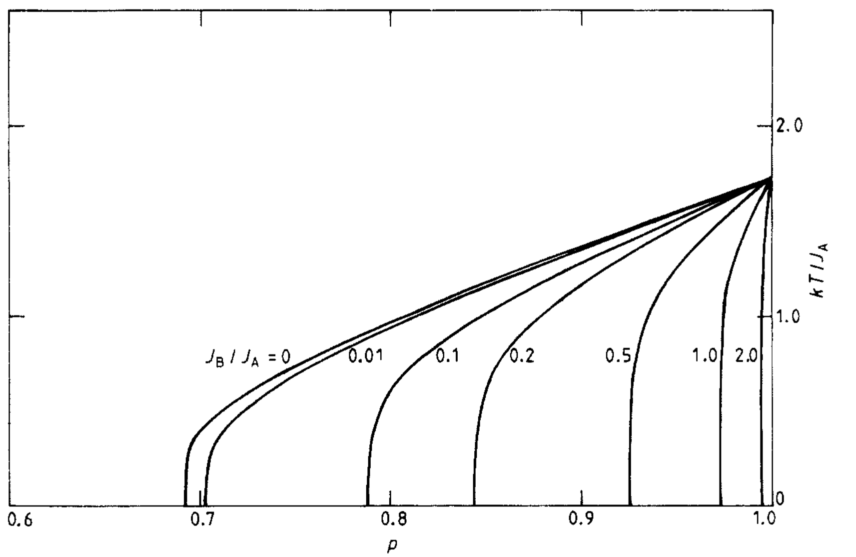

Figure 3. The random-bond Ising model on the square lattice with exchange interaction $J_A>0$ and $-J_B<0$ with respective probabilities p and $1-p$. There exists the spontaneous magnetisation on the low-temperature side of the full curve for each value of the ratios $J_B/J_A=0$, 0.01, 0.1, 0.2, 0.5, 1.0 and 2.0.

where z is the number of the nearest-neighbour lattice sites and N_d is the total number of lattice sites in the dual lattice. For the triangular lattice, we have $N_d = 2N$: N is the total number of lattice sites in the original lattice. Each closed polygon with perimeter b encloses at most $(b^2+12)/48$ lattice sites. Then we have in place of (2.13)

$$\lim_{N\to\infty} \frac{1}{N} \langle\langle N_-\rangle_{0,B_1}\rangle_c \leq \frac{1}{32} \sum_{\substack{n=3\\(n\neq 4)}}^{\infty} \frac{4^n(n^2+3)}{n} \left\langle \left[1+\exp\left(2\beta \sum_{(ij)\in\mathscr{L}(2n)} J_{ij}\right)\right]^{-1} \right\rangle_c . \tag{5.2}$$

For the honeycomb lattice, we have $N_d = N/2$. Each closed polygon with perimeter b encloses at most $b^2/6$ lattice sites. Then we have in place of (2.13)

$$\lim_{N\to\infty} \frac{1}{N} \langle\langle N_-\rangle_{0,B_1}\rangle_c \leq \frac{1}{100} \sum_{b=3}^{\infty} b5^b \left\langle \left[1+\exp\left(2\beta \sum_{(ij)\in\mathscr{L}(b)} J_{ij}\right)\right]^{-1} \right\rangle_c . \tag{5.3}$$

After deriving each inequality corresponding to equation (2.14) from (5.2) or (5.3), we obtain the general conditions for the convergence. When equation (2.16) has a positive solution t_0, we have for $2\beta \geq t_0$

$$\langle \exp(-t_0 J_{ij})\rangle_c < 1/(z-1) \tag{5.4}$$

and for $2\beta \leq t_0$

$$\langle \exp(-2\beta J_{ij})\rangle_c < 1/(z-1). \tag{5.5}$$

Extension of the present results is also possible to random-bond Ising models on the higher-dimensional lattices. Extension to a random-site Ising model will be discussed in a separate article.

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